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Complexity and Tor on a Complete Intersection

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Let $(R, \mathfrak{m})$ be a complete intersection, that is, a local ring whose $\mathfrak{m}$-adic completion is the quotient of a regular local ring by a regular sequence. Suppose M is a finitely generated R-module. It is known that the even and odd Betti sequences of M are eventually given by polynomials of the same degree n; the complexity of M is the nonnegative integer $n + 1$. We use this notion of complexity to study the vanishing of $\operatorname{Tor}_i^R(M, N)$ for finitely generated modules M and N over a complete intersection R. We prove several theorems dealing with rigidity of Tor, which are generalizations and, in certain situations, improvements of known results. The main idea of these rigidity theorems is that the number of consecutive vanishing Tors required in the hypothesis of a rigidity theorem depends more on the minimum of the complexities of M and N rather than on the codimension of R. We give examples showing that this dependence is sharp. We also show that if $M \otimes_R N$ has finite length, then, for sufficiently high indices, two consecutive vanishing Tors force the vanishing of all higher Tors. © 1999 Academic Press

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INTRODUCTION

The celebrated rigidity theorem of Auslander and Lichtenbaum states that if R is a regular local ring (which is a complete intersection of codimension 0) and M and N are finitely generated R-modules, then $\operatorname{Tor}_i^R(M, N) = 0$ for some $i > 0$ implies $\operatorname{Tor}_j^R(M, N) = 0$ for all $j \geq i$. (Auslander proved the unramified case in 1961 [Au] and Lichtenbaum the ramified case in 1966 [L].) It turns out that rigidity of Tor is more mysterious than this for complete intersections of positive codimension: examples of finitely generated modules M and N over a complete intersection R of positive codimension where $\operatorname{Tor}_i^R(M, N) = 0$ but

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$\operatorname{Tor}_{i+1}^R(M, N) \neq 0$ are plentiful (see, e.g., (4.1) of [HW1]). Much of the mystery is undoubtedly due to the fact that modules over a complete intersection of positive codimension no longer necessarily have finite projective dimension. Murthy has shown, however, that if R is a complete intersection of codimension r, then $r + 1$ consecutive vanishing Tors involving a pair of finitely generated R-modules force the vanishing of all subsequent Tors [Mu, (1.6)]. A straightforward induction based on a theorem of Huneke and Wiegand [HW1, (2.4)] shows that the number of consecutive vanishing Tors required in the hypothesis of a rigidity theorem such as Murthy's can be reduced by 1 (to r) provided the tensor product of the pair of modules has finite length, the sum of the dimensions of the modules is strictly less than the dimension of the relevant regular local ring, and the vanishing occurs beyond the dimension of R. This paper serve in part to generalize and, in some cases, improve upon these results of Murthy and Huneke and Wiegand.

A minimal free resolution of a finitely generated module M over a complete intersection R is well behaved in high degree. In particular, the even and odd Betti sequences are eventually given by polynomials of the same degree n. The *complexity* of M, denoted by $\operatorname{cx}_R M$, is the nonnegative integer $n + 1$. For any finitely generated R-module M we have

$$\operatorname{cx}_R M \leq \operatorname{codim} R,$$

where codim R denotes the codimension of R. The abundance of modules of complexity strictly less than codim R shoulders the advantages of the rigidity theorems of this paper, which we now state.

In Section 2 we prove a result (Proposition 2.3) that is analogous to Murthy's theorem: Suppose M and N are finitely generated modules over a complete intersection R. Let c denote the minimum of the complexities of M and N, and let b denote the maximum of all the depths of M and N. If $c + 1$ consecutive Tors vanish anywhere beyond $\dim R - b$, then all Tors beyond $\dim R - b$ vanish.

We also prove a rigidity theorem (Theorem 2.6) that generalizes the result based on the theorem of Huneke and Wiegand: Suppose M and N are finitely generated modules over a complete intersection R, and either M or N has positive complexity c. Let b denote the sum of the depths of M and N. Assume that $M \otimes_R N$ has finite length and $\dim M + \dim N < \dim R + c$. If c consecutive Tors vanish anywhere beyond $\dim R - b$, then all Tors beyond $\dim R - b$ vanish.

They key idea used in the proofs of the rigidity theorems of Section 2 is (Theorem 1.3), the inductive theorem. It allows us to change rings in such a way that the complexities of our modules drop by one if positive and remain zero otherwise.

Also in Section 2, we give a proof of a necessary condition for the vanishing of all higher Tors (Proposition 2.4) due to C. Miller [M]. This necessary condition says that there must exist infinitely many nonzero Tors if the sum of the complexities of M and N is larger than the codimension of R. The reader may wish to refer directly to Theorem 2.1, where some of the results of Section 2 are showcased.

Another point made by this paper is that the vanishing of Tor is somewhat more predictable asymptotically (Theorem 3.1): Suppose M and N are finitely generated modules over a complete intersection R (of arbitrary codimension) such that $M \otimes_R N$ has finite length. Let b denote the maximum of the depths of M and N. Then there exists a positive integer ν satisfying the following: if one even Tor and one odd Tor beyond ν are zero, then all Tors beyond $\dim R - b$ are zero. (The author thanks L. Avramov for suggesting a streamlining of the proof of Theorem 3.1.)

Finally, in Section 4 we give examples, one of which illustrates the sharpness of (Proposition 2.3): There exist complete intersections R with finitely generated maximal Cohen–Macaulay R-modules M and N of equal complexity n such that n consecutive Tors vanish, whereas the first subsequent Tor does not.

0. PRELIMINARIES

Notation and Assumptions

All rings are assumed to be commutative with identity, and all modules are taken to be finitely generated unless otherwise stated. Typically, a *complete intersection* is defined to be a local (meaning also Noetherian) ring $(R, \mathfrak{m}, k)$ whose $\mathfrak{m}$-adic completion is the quotient of a regular local ring $(Q, \mathfrak{n}, k)$ by an ideal $(\mathbf{x})$ generated by a Q-regular sequence $\mathbf{x} = x_1, \ldots, x_r$. However, since all of the mathematical gadgets in this paper are invariant under faithfully flat extension and descent, we will always assume that $R \cong Q/(\mathbf{x})$.

Whenever $(R, \mathfrak{m}, k)$ is a local ring, we define the *codimension* of R, written codim R, to be the nonnegative integer embdim $R - \dim R$, where embdim R denotes $\dim_k \mathfrak{m}/\mathfrak{m}^2$, the *embedding codimension* of R. It turns out [BH, Sect. 2.3] that if R is a complete intersection, the codimension of R is the same as the length of the regular sequence $\mathbf{x}$ defining R, provided $(\mathbf{x}) \subseteq \mathfrak{n}^2$. Hence we will always assume that this is the case.

Suppose R is a complete intersection of codimension $r > 0$. Then a ring R_1 is said to be a *principal lifting of* R if R_1 is a complete intersection of codimension $r - 1$ containing a regular element x such that $R \cong R_1/(x)$.

The Long Exact Sequence

Suppose R is a complete intersection and M and N are R-modules. Whenever R_1 is a principal lifting of R, the standard *change of rings spectral sequence* (see [Rot, (11.64)] or [CE, Chap. XVI, Sect. 5]),

$$\operatorname{Tor}_p^R\left(M, \operatorname{Tor}_q^{R_1}(N, R)\right) \underset{p}{\Rightarrow} \operatorname{Tor}_n^{R_1}(M, N),$$

collapses with respect to one filtration and has zero E^2 terms, except in two rows with respect to the other filtration. Hence, making the identification $\operatorname{Tor}_1^{R_1}(N, R) \cong N$, we obtain the following long exact sequence of Tors:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\mathrm{T}_{i+1}^R & \to & \mathrm{T}_{i+2}^{R_1} & \to & \mathrm{T}_{i+2}^R & \to & \\
\mathrm{T}_i^R & \to & \mathrm{T}_{i+1}^{R_1} & \to & \mathrm{T}_{i+1}^R & \to & \\
\mathrm{T}_{i-1}^R & \to & \mathrm{T}_i^{R_1} & \to & \mathrm{T}_i^R & \to & \\
\vdots & & \vdots & & \vdots & & \\
\mathrm{T}_1^R & \to & \mathrm{T}_2^{R_1} & \to & \mathrm{T}_2^R & \to & \\
\mathrm{T}_0^{R_1} & \to & \mathrm{T}_1^{R_1} & \to & \mathrm{T}_1^R & \to & 0,
\end{array} \tag{0.1}$$

where $\mathrm{T}_i^R := \operatorname{Tor}_i^R(M, N)$ and $\mathrm{T}_i^{R_1} := \operatorname{Tor}_i^{R_1}(M, N)$. This long exact sequence can also be derived without using the spectral sequence machinery [Mu, (1.5)].

Eisenbud Operators

In this section we give a brief account of certain operators originally defined by Eisenbud [E]. One can find in [A1, Sect. 1] an expanded treatment, including analogous operators defined by Avramov on injective complexes, and induced cohomology operators.

Let R be a Noetherian ring of the form $R = Q/(\mathbf{x})$, where $\mathbf{x} = x_1, \ldots, x_r$ is a Q-regular sequence. Eisenbud defined degree -2 endomorphisms $t_j = t_j(Q, \mathbf{x}, \mathbf{F})\colon \mathbf{F} \to \mathbf{F}$ of a complex $\mathbf{F}\colon \cdots \xrightarrow{\partial} F_1 \xrightarrow{\partial} F_0$ of free R-modules as follows. Let

$$\tilde{\mathbf{F}}\colon \cdots \to \tilde{F}_{i+2} \xrightarrow{\tilde{\partial}} \tilde{F}_{i+1} \xrightarrow{\tilde{\partial}} \tilde{F}_i \to \cdots$$

be a sequence of free Q-modules with maps $\tilde{\partial}$ between them so that $\mathbf{F} = R \otimes_Q \tilde{\mathbf{F}}$. Since $\tilde{\partial}^2 \equiv 0 \pmod{(\mathbf{x})}$, we can write $\tilde{\partial}^2 = \sum_{j=1}^r x_j \tilde{t}_j$, where $\tilde{t}_j\colon \tilde{\mathbf{F}} \to \tilde{\mathbf{F}}$ is a degree -2 endomorphism of the graded Q-module $\tilde{\mathbf{F}}$ for all j

$(1 \le j \le r)$. Now define $t_j = t_j(Q, \mathbf{x}, \mathbf{F})\colon \mathbf{F} \to \mathbf{F}$ by $t_j = R \otimes \tilde{t}_j$. Then, as is shown in [E], $t_j(Q, \mathbf{x}, \mathbf{F})$ is a degree -2 endomorphism of the complex $\mathbf{F}$. One could write $t_j(Q, \mathbf{x}, \tilde{\mathbf{F}}, \tilde{\partial})$ instead of $t_j(Q, \mathbf{x}, \mathbf{F})$ to express the dependence of the definition on the liftings $\tilde{\mathbf{F}}$ and $\tilde{\partial}$. However [E, (1.3)] or [A1, (1.3)], if $\mathbf{F} \xrightarrow{f} \mathbf{G}$ is any homomorphism of complexes, then $ft_j(Q, \mathbf{x}, \mathbf{F})$ is homotopic to $t_j(Q, \mathbf{x}, \mathbf{G})f$, so that any two such definitions of t_j agree up to homotopy. Hence we shall suppress this dependence on the lifting and simply write $t_j(Q, \mathbf{x}, \mathbf{F})$.

We can now state Eisenbud's theorem, which is used in the next section:

THEOREM 0.2 [E, (3.1)]. *Let Q be a regular local ring with infinite residue field, and let I be an ideal of Q generated by a Q-regular sequence. Set $R = Q/I$. If $\mathbf{F}\colon \cdots \to F_1 \to F_0$ is the minimal R-free resolution of a finitely generated R-module, then there exists a Q-regular sequence $\mathbf{x} = x_1, \dots, x_r$ generating I such that*

$$t_1(Q, \mathbf{x}, \mathbf{F})\colon F_{i+2} \to F_i$$

is surjective for sufficiently large i.

1. COMPLEXITY

In this section we define complexity, which is a homological characteristic of modules first introduced by Alperin and Evens [AE] in the setting of group representations and group cohomology. It was then brought into local ring theory by Avramov in [A1]. We then state some known theorems that are used in the proof of Theorem (1.3), the main result of this section.

DEFINITION. Let $(R, \mathfrak{m}, k)$ be local ring, and let $b_n^R(M)$ denote the nth Betti number of the R-module M (i.e., $b_n^R(M) := \dim_k \operatorname{Tor}_n^R(M, k)$). We define the *complexity* of M to be

$$\operatorname{cx}_R M := \inf\left\{c \in \mathbb{N} \mid \exists \gamma \in \mathbb{R} \text{ such that } b_n^R(M) \le \gamma n^{c-1} \text{ for } n \gg 0\right\}.$$

Note that $\operatorname{cx}_R M = 0$ if and only if M has finite projective dimension over R. Modules over arbitrary local rings may have infinite complexity (see, e.g., [A2] and [S]). However, for complete intersections, the complexity is always finite. In fact, we have the following result, part (1) of which is due to Gulliksen [G], and part (2) of which is due to Avramov [A1]; see the proof of (Example 4.1):

THEOREM 1.1. *Let R be a complete intersection of codimension r and M an R-module.*

(1) *There exist polynomials* $p_e(t)$ *and* $p_o(t)$ *in* $\mathbb{Q}[t]$ *both of degree* $\leq r-1$ *such that*

$$p_e(n) = b^R_{2n}(M) \quad \text{and} \quad p_o(n) = b^R_{2n+1}(M) \qquad \text{for all } n \gg 0.$$

(2) *Moreover*, $\deg p_e = \deg p_o$.

(By convention, the degree of the zero polynomial is -1.)

Hence, for modules M over a complete intersection, the complexity of M is just one more than the degree of these polynomials, and we have

$$\operatorname{cx}_R M \leq \operatorname{codim} R.$$

Remark. The inequality $\operatorname{cx}_R M \leq \operatorname{codim} R$ can also easily be deduced from the earlier work of Shamash [Sh].

For graded modules X and Y over a graded ring A, we denote by ${}^*\operatorname{Hom}_A(X,Y)$ the graded A-module $\oplus_{i\in\mathbb{Z}} \operatorname{Hom}_i(X,Y)$, where $\operatorname{Hom}_i(X,Y)$ denotes the set of homogeneous A-homomorphisms of degree i from X to Y. The following result from [E] concerns the *divided power algebra* $\mathbf{D}$, which is defined to be

$$\mathbf{D} := {}^*\operatorname{Hom}_R(R[t], R),$$

where the variable t is assigned degree -2 and R is considered a graded R-module concentrated in degree zero. Hence $\mathbf{D} = \oplus_{n\in\mathbb{Z}} D_n$ is a graded R-module with homogeneous components,

$$D_n \cong \begin{cases} R & \text{if } n \text{ is nonnegative and even,} \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathbf{G}$ is any complex of Q-modules, with $Q \to R$ a ring homomorphism, we will write $\mathbf{D} \otimes_Q \mathbf{G}$ to denote the graded tensor product of $\mathbf{D}$ and $\mathbf{G}$, where we regard $\mathbf{G}$ as a graded Q-module. Eisenbud [E, (7.2)] defines a differential so that $\mathbf{D} \otimes_Q \mathbf{G}$ is in fact a complex of R-modules, but we shall not be concerned with the differential in our application of the next theorem.

THEOREM 1.2 [E, (8.1)]. *Let* R_1 *be a local ring, and let* $x \in R_1$ *be a nonzero divisor. Set* $R = R_1/(x)$, *and let* $\mathbf{F}\colon \cdots \xrightarrow{\partial} F_1 \xrightarrow{\partial} F_0$ *be an* R*-free resolution of an* R*-module* M *such that* $t(R_1, x, \mathbf{F})\colon \mathbf{F} \to \mathbf{F}$ *is surjective. Let* $(\mathbf{F}^\#, \partial^\#)$ *be any lifting of* $(\mathbf{F}, \partial)$, *that is, let*

$$\mathbf{F}^\#\colon \cdots \xrightarrow{\partial^\#} F_1^\# \xrightarrow{\partial^\#} F_0^\#$$

be a sequence of maps of free R_1-modules such that $R \otimes_{R_1} \mathbf{F}^\# = \mathbf{F}$. Suppose $t^\#\colon \mathbf{F}^\# \to \mathbf{F}^\#$ is the degree -2 endomorphism of $\mathbf{F}^\#$ satisfying $\partial^\# = xt^\#$. Then

$$\mathbf{K}^\# := \ker(t^\#)\colon \cdots \xrightarrow{\partial^\#|K_3^\#} K_2^\# \xrightarrow{\partial^\#|K_2^\#} F_1^\# \xrightarrow{\partial^\#} F_0^\#$$

is an R_1-free resolution of M.

Moreover, letting $\mathbf{D}$ be the divided power algebra, we have $\mathbf{F} \cong \mathbf{D} \otimes_{R_1} \mathbf{K}^\#$ as complexes.

Remark. If $\mathbf{F}$ is a minimal resolution, so is $\mathbf{K}^\#$.

Now we come to the main result of this section.

THEOREM 1.3 (the inductive theorem). *Let R be a complete intersection of codimension $r > 0$ with infinite residue field. Suppose $\mathscr{F}$ is a finite set of R-modules. Then there exists a principal lifting R_1 of R such that, for all $M \in \mathscr{F}$,*

$$\operatorname{cx}_{R_1} M = \begin{cases} \operatorname{cx}_R M - 1 & \text{if } \operatorname{cx}_R M > 0 \\ 0 & \text{if } \operatorname{cx}_R M = 0. \end{cases}$$

Proof. Write $R = Q/I$, where Q is a regular local ring and I is generated by a regular sequence of length r. For each $M \in \mathscr{F}$, choose a minimal resolution $\mathbf{F}$ of M. By applying (0.2) to the direct sum of all of the modules in $\mathscr{F}$, we obtain a generating set $\mathbf{x} = x_1, \ldots, x_r$ of I such that $t_1(Q, \mathbf{x}, \mathbf{F})$ is eventually surjective for all $M \in \mathscr{F}$. That is, there exists $n \in \mathbb{N}$ such that

$$t_1(Q, \mathbf{x}, \mathbf{F})\colon F_{i+2} \to F_i$$

is onto for all $i \geq n$ and for all of the resolutions $\mathbf{F}$.

Now we restrict our attention to a single $M \in \mathscr{F}$ with minimal resolution $\mathbf{F}$. Let $x = x_1$ and $R_1 = Q/(x_2, \ldots, x_r)$ and write $t(R_1, x, \mathbf{F}) = t_1(R_1, x_1, \mathbf{F})$. It is clear from the construction of the Eisenbud operators that $t_1(Q, \mathbf{x}, \mathbf{G})$ is homotopic to $t_1(R_1, x_1, \mathbf{G})$ for any complex of free R-modules $\mathbf{G}$. Hence $t(R_1, x, \mathbf{F})\colon F_{i+2} \to F_i$ is surjective for all $i \geq n$. Set $S_n^M := \operatorname{syz}_n^R(M)$ and $\mathbf{F}_{\geq n}\colon \cdots \xrightarrow{\partial} F_{n+1} \xrightarrow{\partial} F_n$. Define $t'\colon \mathbf{F}_{\geq n} \to \mathbf{F}_{\geq n}$ by $t'(a) = t(R_1, x, \mathbf{F})(a)$ if $a \in F_i$ with $i \geq n+2$ and $t'(a) = 0$ otherwise. Then t' is obviously surjective and homotopic to $t(R_1, x, \mathbf{F}_{\geq n})$. Therefore

$$t(R_1, x, \mathbf{F}_{\geq n})\colon \mathbf{F}_{\geq n} \to \mathbf{F}_{\geq n} \text{ is surjective.}$$

Take any lifting $(\mathbf{F}^\#_{\geq n}, \partial^\#_{\geq n})$ of $(\mathbf{F}_{\geq n}, \partial_{\geq n})$, i.e., $\mathbf{F}_{\geq n} = R \otimes_{R_1} \mathbf{F}^\#_{\geq n}$, and let $t^\#$ be the degree -2 endomorphism of $\mathbf{F}^\#_{\geq n}$ satisfying $(\partial^\#_{\geq n})^2 = xt^\#$. Then by (1.2),

$$\mathbf{K}^\# := \ker(t^\#)\colon \cdots \xrightarrow{\partial^\#_{\geq n}|K_3^\#} K_2^\# \xrightarrow{\partial^\#_{\geq n}|K_2^\#} F_{n+1}^\# \xrightarrow{\partial^\#_{\geq n}} F_n^\#$$

is a minimal resolution of S_n^M over R_1 and

$$\mathbf{F}_{\geq n} \cong \mathbf{D} \otimes_{R_1} \mathbf{K}^{\#}$$

as complexes. One can now write down the free modules in $\mathbf{F}_{\geq n}$ explicitly in terms of the $K_i^{\#}$. Because all of the odd homogeneous components of the graded R-module $\mathbf{D}$ are zero, it is convenient to write the even and odd components of $\mathbf{F}$ independently.

For all $i \geq 0$,

$$F_{n+2i} = \bigoplus_{u+v=2i} D_u \otimes_{R_1} K_v^{\#} \cong \bigoplus_{s=0}^{i} D_{2s} \otimes_{R_1} K_{2(i-s)}^{\#} \cong \bigoplus_{s=0}^{i} K_{2s}$$

and

$$F_{n+2i+1} = \bigoplus_{u+v=2i+1} D_u \otimes_{R_1} K_v^{\#} \cong \bigoplus_{s=0}^{i} D_{2s} \otimes_{R_1} K_{2(i-s)+1}^{\#} \cong \bigoplus_{s=0}^{i} K_{2s+1},$$

where $K_0^{\#} := F_n^{\#}$, $K_1^{\#} := F_{n+1}^{\#}$ and $K_j := R \otimes_{R_1} K_j^{\#}$ for all j.

By Theorem 1.1, there are polynomials $p_e^{\#}$ and $p_o^{\#}$ such that

$$p_e^{\#}(i) = b_{2i}^{R_1}(S_n^M) = \operatorname{rank}(K_{2i}) \quad \text{and}$$
$$p_o^{\#}(i) = b_{2i+1}^{R_1}(S_n^M) = \operatorname{rank}(K_{2i+1})$$

for all $i \geq N$, say, and $d := \deg p_e^{\#} = \deg p_o^{\#} \leq \operatorname{codim} R_1$. Hence for $i \gg 0$,

$$b_{2i}^{R}(S_n^M) = \operatorname{rank} F_{n+2i} = \sum_{s=0}^{i} \operatorname{rank} K_{2s} = \sum_{s=N}^{i} p_e^{\#}(s) + \sum_{s=0}^{N-1} b_{2s}^{R_1}(S_n^M)$$

and

$$b_{2i+1}^{R}(S_n^M) = \operatorname{rank} F_{n+2i+1} = \sum_{s=0}^{i} \operatorname{rank} K_{2s+1}$$
$$= \sum_{s=N}^{i} p_o^{\#}(s) + \sum_{s=0}^{N-1} b_{2s+1}^{R_1}(S_n^M).$$

If $\operatorname{cx}_R M = 0$, then $\operatorname{cx}_R S_n^M = 0$, and so [K, Exercise 4, p. 205] $\operatorname{cx}_{R_1} M = \operatorname{cx}_{R_1} S_n^M = 0$. (Note that $\operatorname{cx}_{R_1} M = \operatorname{cx}_{R_1} S_n^M$ as $\operatorname{Tor}_{i+j}^{R_1}(M, N) \cong \operatorname{Tor}_i^{R_1}(S_j^M, N)$ for all N, $i \geq 2$ and $j \geq 0$.) If $\operatorname{cx}_R M > 0$, then $\operatorname{cx}_R S_n^M > 0$, and the polynomials

$$p_e(t) := \sum_{s=N}^{t} p_e^{\#}(s) + \sum_{s=0}^{N-1} b_{2s}^{R_1}(S_n^M)$$

and

$$p_o(t) := \sum_{s=N}^{t} p_o^{\#}(s) + \sum_{s=0}^{N-1} b_{2s+1}^{R_1}(S_n^M)$$

in $\mathbb{Q}[t]$ are both of degree $d+1$. (This follows from the fact that $A_d(t) := \sum_{s=N}^{t} s^d$ is a polynomial of degree $d+1$.) Hence, $\mathrm{cx}_{R_1} M = \mathrm{cx}_{R_1} S_n^M = d = \mathrm{cx}_R S_n^M - 1 = \mathrm{cx}_R M - 1$, and the proof is complete. ∎

Remarks. (1) The technique of the proof of Theorem 1.3 can be used to give an alternative proof of [AGP, (8.1)] in the case of a complete intersection.

(2) L. Avramov has pointed out to me that Theorem 1.3 follows from (3.2.3) and (3.6) of [A1].

2. VANISHING OF TOR

We first record some of the results of this section in the following theorem.

THEOREM 2.1. *Suppose that R is a d-dimensional complete intersection of codimension $r > 0$. Let M and N be R-modules, and set* $b := \max\{\operatorname{depth} M, \operatorname{depth} N\}$ *and* $c := \min\{\mathrm{cx}_R M, \mathrm{cx}_R N\}$. *The following are equivalent*:

(1) $\mathrm{Tor}_i^R(M,N) = 0$ *for all* $i \gg 0$.

(2) $\mathrm{Tor}_i^R(M,N) = 0$ *for all* $i \geq d - b + 1$.

(3) *For some* $i \geq d - b + 1$,

$$\mathrm{Tor}_i^R(M,N) = \cdots = \mathrm{Tor}_{i+c}^R(M,N) = 0,$$

and $\mathrm{cx}_R M + \mathrm{cx}_R N \leq r$.

The proof of Theorem 2.1 will follow from the next three propositions, 2.2, 2.3, and 2.4. Proposition 2.2 is probably known, but I have not seen it in the literature:

PROPOSITION 2.2. *Let R be complete intersection of dimension d, and let M and N be R-modules. Set* $b := \max\{\operatorname{depth} M, \operatorname{depth} N\}$. *If* $\mathrm{Tor}_i^R(M,N) = 0$ *for all* $i \gg 0$, *then* $\mathrm{Tor}_i^R(M,N) = 0$ *for all* $i \geq d - b + 1$.

Proof. We use induction on the codimension r of R.

$r = 0$: In this case R is a regular local ring, so that the Auslander–Buchsbaum formula [BH, (1.3.3)] gives the desired result.

$r > 0$: Choose any principle lifting R_1 of R. We then have the long exact sequence of Tors (0.1). Since $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \gg 0$, we have $\operatorname{Tor}_i^{R_1}(M, N) = 0$ for all $i \gg 0$. By induction, $\operatorname{Tor}_i^{R_1}(M, N) = 0$ for all $i \geq (d+1) - b + 1$. Hence, by (0.1), we see that $\operatorname{Tor}_i^R(M, N) \cong \operatorname{Tor}_{i+2}^R(M, N)$ for all $i \geq d - b + 1$. Since $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \gg 0$, we conclude that $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq d - b + 1$, as advertised. ∎

The next proposition, which is analogous to Murthy's rigidity result, illustrates the phenomenon that if M is a module over a complete intersection R of codimension r with $c := \operatorname{cx}_R M < r$, then M behaves homologically like a module over a complete intersection of codimension c.

PROPOSITION 2.3. *Suppose R is a complete intersection of dimension d, and let M and N be R-modules with $c := \min\{\operatorname{cx}_R M, \operatorname{cx}_R N\}$ and $b :=$* $\max\{\operatorname{depth} M, \operatorname{depth} N\}$. *If*

$$\operatorname{Tor}_i^R(M, N) = \cdots = \operatorname{Tor}_{i+c}^R(M, N) = 0$$

for some $i \geq d - b + 1$, then $\operatorname{Tor}_j^R(M, N) = 0$ *for all $j \geq d - b + 1$.*

Proof. First we reduce to the case where R has infinite residue field. Suppose the residue field of $(R, \mathfrak{m})$ is finite. If z is an indeterminate over R, then $A = R[z]_{\mathfrak{m}R[z]}$ is a complete intersection of codimension r and dimension d with infinite residue field such that the ring extension $R \hookrightarrow A$ is faithfully flat. By faithfully flat descent [EGA2, (2.5.8)] we have

$$\operatorname{Tor}_i^A(M \otimes A, N \otimes A) = 0 \qquad \text{if and only if} \quad \operatorname{Tor}_i^R(M, N) = 0,$$

$\operatorname{depth}_A M \otimes_R A = \operatorname{depth}_R M$, and $\operatorname{cx}_A(M \otimes_R A) = \operatorname{cx}_R M$. Thus, we may assume that R has infinite residue field.

Without loss of generality suppose that $c = \operatorname{cx}_R M$. We proceed by induction on c.

$c = 0$: In this case, $\operatorname{pd}_R M < \infty$ so that $\operatorname{Tor}_j^R(M, N) = 0$ for all $j \gg 0$. Now (2.2) says that $\operatorname{Tor}_j^R(M, N) = 0$ for all $j \geq d - b + 1$.

$c > 0$: Here we use (1.3) to find a principal lifting R_1 of R such that $\operatorname{cx}_{R_1} M = c - 1$. Looking at (0.1), we see that if

$$\operatorname{Tor}_i^R(M, N) = \cdots = \operatorname{Tor}_{i+c}^R(M, N) = 0$$

for some $i \geq d - b + 1$, then

$$\operatorname{Tor}_{i+1}^{R_1}(M, N) = \cdots = \operatorname{Tor}_{i+c}^{R_1}(M, N) = 0;$$

and $i+1 \geq (d+1) - b + 1$. Hence, by induction, we get that $\operatorname{Tor}_j^{R_1}(M, N) = 0$ for all $j \geq d - b + 2$. Another glance at (0.1) shows that

$$\operatorname{Tor}_j^R(M, N) = 0 \qquad \text{for all } j \geq d - b + 1,$$

as desired. ∎

Huneke and Wiegand have recently proved [HW2, (1.9)] that if R is a hypersurface and M and N are two R-modules such that $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \gg 0$, then either M or N has finite projective dimension over R. The next proposition, due to Claudia Miller, is a generalization of this result. The proof we give depends on the Huneke–Wiegand result for hypersurfaces and uses techniques different from those of Miller's proof (which does not depend on the Huneke–Wiegand result).

PROPOSITION 2.4 (C. Miller). *Suppose $(R, \mathfrak{m}, k)$ is a complete intersection of codimension $r > 0$ and M and N are R-modules. If $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \gg 0$, then $\operatorname{cx}_R M + \operatorname{cx}_R N \leq r$.*

Proof. Without loss of generality, we may assume that R has infinite residue field. If the residue field k of R is not algebraically closed, there is a faithfully flat extension $R \hookrightarrow A$, where A is a complete intersection of the same codimension as R and such that the residue field of A is algebraically closed [EGA1, (6.6.1) and (6.8.2)]. Hence we may also assume that k is algebraically closed.

If $r = 1$, the theorem is just (1.9) of [HW2]; therefore, we will assume $r > 1$. Let $\mathbf{x} = x_1, \ldots, x_r$ be a regular sequence in the regular local ring Q defining R, and let $\mathfrak{n}$ denote the maximal ideal of Q. For any R-module M there is an associated affine algebraic set containing the origin (a subset of the r-dimensional k-vector space $(\mathbf{x})/\mathfrak{n}(\mathbf{x})$) defined by Avramov [A1, (3.8)] and denoted $V(Q, \mathbf{x}, M)$. Avramov proves [A1, (3.12)] that the dimension of $V(Q, \mathbf{x}, M)$ is equal to $\operatorname{cx}_R M$. For any $z \in (\mathbf{x})$, let $\bar{z}$ denote the image of z in $(\mathbf{x})/\mathfrak{n}(\mathbf{x})$.

Suppose that $\operatorname{cx}_R M + \operatorname{cx}_R N > r$. Then by basic intersection theory [Ha, I (7.1)], $V(Q, \mathbf{x}, M) \cap V(Q, \mathbf{x}, N)$ has positive dimension. Hence there exists $z \in (\mathbf{x})$ such that $0 \neq \bar{z} \in V(Q, \mathbf{x}, M) \cap V(Q, \mathbf{x}, N)$. Note that since $\bar{z} \neq 0$, z is a nonzero divisor of Q in $(\mathbf{x})$.

Corollary (3.11) of [A1] states (in the case of a regular local ring Q): Suppose z is a nonzero divisor of Q in $(\mathbf{x})$. Then $\bar{z} \in V(Q, \mathbf{x}, M)$ if and only if $\operatorname{pd}_{Q/(z)} M = \infty$. Hence we conclude that $\operatorname{pd}_{Q/(z)} M = \operatorname{pd}_{Q/(z)} N = \infty$. Since $Q/(z)$ is a hypersurface, the Huneke–Wiegand result implies that there exist infinitely many nonzero $\operatorname{Tor}_i^{Q/(z)}(M, N)$. As z is part of a minimal generating set of $(\mathbf{x})$, $r - 1$ applications of (0.1) show there must also exist infinitely many nonzero $\operatorname{Tor}_i^R(M, N)$. ∎

The following lemma is a generalization of Proposition 2.2 of [HW1].

LEMMA 2.5. *Let R be a complete intersection of dimension d and codimension $r > 0$. Suppose M and N are R-modules both with complexity at most* 1 *over R. Let $b :=$* depth M + depth N. *Assume*

(1) *$M \otimes_R N$ has finite length.*

(2) $\dim M + \dim N < d + 1$.

Then $l(\mathrm{Tor}_i^R(M, N)) = l(\mathrm{Tor}_{d-b+1}^R(M, N))$ *for all $i \geq d - b + 1$.*

Proof. Again we can assume that R has infinite residue field. (Length as well as dimension are preserved under the faithfully flat extension $R \to R[z]_{\mathfrak{m} R[z]}$.) We use Theorem 1.3 to find a principal lifting R_1 of R over which $\mathrm{cx}_{R_1} M = \mathrm{cx}_{R_1} N = 0$. That is, regarded as modules over R_1, both M and N have finite projective dimension. This, together with the facts that $M \otimes_R N$ has finite length and $\dim M + \dim N < d + 1$, allows us to use Roberts' theorem [R] to conclude that $\chi^{R_1}(M, N) = 0$ (where $\chi^{R_1}(M, N) := \sum_{i \geq 0} (-1)^i l(\mathrm{Tor}_i^{R_1}(M, N))$ is the *Euler characteristic*). Also, since $M \otimes_R N$ has finite length, depth $\mathrm{Tor}_i^{R_1}(M, N) = 0$ for all $i \geq 0$. Therefore, by Theorem 1.2 of [Au], $\mathrm{Tor}_i^{R_1}(M, N) = 0$ for all $i >$ depth R_1 − depth M − depth $N = d - b + 1$. Now (0.1) looks like

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \\
\mathrm{T}^R_{d-b+2} & \to & 0 & \to & \mathrm{T}^R_{d-b+3} & \to & \\
\mathrm{T}^R_{d-b+1} & \to & 0 & \to & \mathrm{T}^R_{d-b+2} & \to & \\
\mathrm{T}^R_{d-b} & \to & \mathrm{T}^{R_1}_{d-b+1} & \to & \mathrm{T}^R_{d-b+1} & \to & \\
\vdots & & \vdots & & \vdots & & \\
\mathrm{T}^R_1 & \to & \mathrm{T}^{R_1}_2 & \to & \mathrm{T}^R_2 & \to & \\
\mathrm{T}^{R_1}_0 & \to & \mathrm{T}^{R_1}_1 & \to & \mathrm{T}^R_1 & \to & 0.
\end{array}
\tag{2.5.1}
$$

Since the alternating sum of the lengths of the Tors in the exact sequence $0 \to \mathrm{T}^R_{d-b+2} \to \cdots \to \mathrm{T}^R_1 \to 0$ is zero, and since $\chi^{R_1}(M, N) = 0$, one concludes that $l(\mathrm{T}^R_{d-b+1}) = l(\mathrm{T}^R_{d-b+2})$. Finally, since $\mathrm{T}^R_i \cong \mathrm{T}^R_{i+2}$ for all $i \geq d - b + 1$, the result follows. ∎

As a consequence of this, we have the following vanishing theorem, which is another example of the phenomenon alluded to before Proposition 2.3. This theorem reduces by one the number of consecutive vanishing Tors required in the hypotheses. However, the trade-off is that one must make additional assumptions on the modules.

The author thanks Petra Constapel for a suggestion that resulted in a sharpening of Lemma 2.5 and Theorem 2.6.

THEOREM 2.6. *Let R be a complete intersection of dimension d and suppose M and N are R-modules. Let* $c := \max\{\operatorname{cx}_R M, \operatorname{cx}_R N, 1\}$ *and* $b :=$ depth M + depth N. *Assume that*

(1) *$M \otimes_R N$ has finite length.*

(2) $\dim M + \dim N < d + c$.

If $\operatorname{Tor}_i^R(M,N) = \cdots = \operatorname{Tor}_{i+c-1}^R(M,N) = 0$ *for some* $i \geq d - b + 1$, *then* $\operatorname{Tor}_j^R(M,N) = 0$ *for all* $j \geq d - b + 1$.

Proof. Again assuming the residue field is infinite, we proceed by induction on c.

$c = 1$: In this case we can apply Lemma 2.5 to obtain $l(\operatorname{Tor}_j^R(M,N)) = l(\operatorname{Tor}_{d-b+1}^R(M,N))$ for all $j \geq d - b + 1$.

$c > 1$: We again use Theorem 1.3 to find a principal lifting R_1 of R over which the complexities of M and N drop by 1 if positive and remain 0 otherwise. Since either M or N must have complexity greater than 1 over R, we have

$$\max\{\operatorname{cx}_{R_1} M, \operatorname{cx}_{R_1} N, 1\} = c - 1.$$

Now if $\operatorname{Tor}_i^R(M,N) = \cdots = \operatorname{Tor}_{i+c-1}^R(M,N) = 0$ for some $i \geq d - b + 1$, one sees from (0.1) that

$$\operatorname{Tor}_{i+1}^{R_1}(M,N) = \cdots = \operatorname{Tor}_{i+c-1}^{R_1}(M,N) = 0.$$

Also, $i + 1 \geq (d+1) - b + 1$ and $\dim M + \dim N < d + c = (d+1) + (c-1)$. Hence, by induction, we conclude that $\operatorname{Tor}_j^{R_1}(M,N) = 0$ for all $j \geq (d+1) - b + 1$. Another look at (0.1) confirms that $\operatorname{Tor}_j^R(M,N) = 0$ for all $j \geq d - b + 1$, as desired. ∎

It is well known that modules of complexity 1 over a complete intersection R have eventually periodic resolutions of period ≤ 2 [E], with periodicity beginning no later than the $(\dim R - \operatorname{depth} M + 1)$st stage. The following proposition shows that the sequence of Tors involving a module M of complexity 1 may become periodic earlier than does the minimal resolution of M. For instance, one could take a module M of complexity 1 and depth 0 and a module N of complexity $c \geq 2$ and depth d over a complete intersection R of codimension $\geq c$. (Such modules exist [AGP, (5.8)].) Then $\operatorname{Tor}_i^R(M,N) \cong \operatorname{Tor}_{i+2}^R(M,N)$ for all $i \geq 1$, even though the resolution of N is never periodic and the resolution of M does not become periodic until the $(d+1)$st stage.

PROPOSITION 2.7. *Let M and N be modules over a d-dimensional complete intersection R such that either M or N has complexity 1 over R. Let*

$b := \max\{\operatorname{depth} M, \operatorname{depth} N\}$. *Then* $\operatorname{Tor}_i^R(M, N) \cong \operatorname{Tor}_{i+2}^R(M, N)$ *for all* $i \geq d - b + 1$.

Proof. We may assume that the residue field of R is infinite. Suppose, without loss of generality, that $\operatorname{cx}_R M = 1$. Choose a principal lifting R_1 of R such that $\operatorname{cx}_{R_1} M = 0$. Then $\operatorname{Tor}_i^{R_1}(M, N) = 0$ for $i \gg 0$. Hence, by (2.2), $\operatorname{Tor}_i^{R_1}(M, N) = 0$ for $i \geq d - b + 2$. Looking at (0.1), we see that $\operatorname{Tor}_i^R(M, N) \cong \operatorname{Tor}_{i+2}^R(M, N)$ for all $i \geq d - b + 1$. ∎

The following theorem is a variation of Proposition 2.3 when one of the modules has complexity 2.

PROPOSITION 2.8. *Let R be a d-dimensional complete intersection and let M and N be R-modules. Suppose $\operatorname{cx}_R M \leq 2$, and set $b := \max\{\operatorname{depth} M, \operatorname{depth} N\}$. If $\operatorname{Tor}_i^R(M, N) = \operatorname{Tor}_{i+1}^R(M, N) = 0$ for some $i \geq d - b + 1$ and $\operatorname{Tor}_{i+2n}^R(M, N) = 0$ for some $n \geq 1$, then $\operatorname{Tor}_j^R(M, N) = 0$ for all $j \geq d - b + 1$.*

Proof. By Proposition 2.3 we may assume that $\operatorname{cx}_R M = 2$. We first reduce to the case where $\operatorname{Tor}_j^R(M, N)$ has finite length for all $j \geq i$. That is, assuming the theorem holds when, in addition, $\operatorname{Tor}_j^R(M, N)$ has finite length for all $j \geq i$, we prove the theorem holds in general. We proceed by induction on d.

$d = 0$: In this case all of the Tor modules have finite length, so we are done.

$d > 0$: Let p be any nonmaximal prime ideal of R. Then we have $\dim R_p < \dim R$, $\operatorname{Tor}_i^{R_p}(M_p, N_p) = \operatorname{Tor}_{i+1}^{R_p}(M_p, N_p) = \operatorname{Tor}_{i+2n}^{R_p}(M_p, N_p) = 0$, and $\operatorname{cx}_{R_p} M_p \leq 2$. Let $d_p := \dim R_p$, and likewise for b_p. Then we have $d - b + 1 \geq d_p - b_p + 1$. (This inequality can be seen as follows. Suppose $\operatorname{depth} M \geq \operatorname{depth} N$. Take an exact sequence $0 \to S_{d-b} \to F_{d-b-1} \to \cdots \to F_0 \to M \to 0$, where S_{d-b} is a $(d-b)$th syzygy module of M. By the Depth Lemma, S_{d-b} is maximal Cohen–Macaulay. Localizing this exact sequence at p, we get an exact sequence $0 \to (S_{d-b})_p \to (F_{d-b-1})_p \to \cdots \to (F_0)_p \to M_p \to 0$. Note that $(S_{d-b})_p$ is maximal Cohen–Macaulay since S_{d-b} is. Using the Depth Lemma on the localized exact sequence, we conclude that $d - b \geq d_p - \operatorname{depth} M_p \geq d_p - b_p$.) Hence $i \geq d_p - b_p + 1$, and by induction we get that $\operatorname{Tor}_j^{R_p}(M_p, N_p) = 0$ for all $j \geq d_p - b_p + 1$. Since this holds for all nonmaximal primes, we conclude that $\operatorname{Tor}_j^R(M, N)$ has finite length for all $j \geq i$. Hence we are done by the finite length case.

Finally, we prove the theorem assuming that $\operatorname{Tor}_j^R(M, N)$ has finite length for all $j \geq i$. We also assume that the residue field of R is infinite. Choose a principal lifting R_1 of R such that $\operatorname{cx}_{R_1} M = 1$. Now look at (0.1). Since $T_i^R = T_{i+1}^R = 0$, we get $T_{i+1}^{R_1} = 0$. But since $\operatorname{cx}_{R_1} M = 1$, $T_j^{R_1} \cong T_{j+2}^{R_1}$

for all $j \geq d - b + 2$ (Proposition 2.7), and we conclude that $\mathrm{T}^{R_1}_{i+1+2\ell} = 0$ for all $\ell \geq 0$. Hence $\mathrm{T}^{R}_{i+1+2\ell} = 0$ for all $\ell \geq 0$. Now (0.1) says that $\mathrm{T}^{R_1}_{i+2} \cong \mathrm{T}^{R}_{i+2}$, and we have short exact sequences

$$0 \to \mathrm{T}^{R_1}_{i+2+2\ell} \to \mathrm{T}^{R}_{i+2+2\ell} \to \mathrm{T}^{R}_{i+2\ell} \to 0$$

for all $\ell \geq 1$. If we let t^R_j denote the length of T^R_j, and define $t^{R_1}_j$ similarly, these short exact sequences yield the equation

$$t^R_{i+2+2\ell} = (\ell + 1)t^R_{i+2}$$

for all $\ell \geq 0$. Finally, since $\mathrm{T}^R_{i+2n} = 0$ for some $n \geq 1$, we get $\mathrm{T}^R_{i+2} = 0$. Therefore, $\mathrm{T}^R_j = 0$ for all $j \gg 0$ by Proposition 2.3, which implies that $\mathrm{T}^R_j = 0$ for all $j \geq d - b + 1$ by Proposition 2.2 ∎

Remark. As is apparent from the bound $d - b + 1$, the vanishing theorems of this section have no content when either M or N has finite projective dimension. Understanding the vanishing of $\mathrm{Tor}^R_i(M, N)$ for $i < d - b + 1$ is perhaps tantamount to understanding rigidity of modules of finite projective dimension.

3. ASYMPTOTIC RIGIDITY OF TOR

In this section we prove a theorem which shows that vanishing of Tor in higher degree is often more rigid than the vanishing of Tor in low degree:

THEOREM 3.1. *Let $(R, \mathfrak{m}, k)$ be a complete intersection of dimension d. Let M and N be R-modules such that $M \otimes_R N$ has finite length. Set $b :=$* $\max\{\operatorname{depth} M, \operatorname{depth} N\}$. *Then there exists a positive integer ν such that if* $\mathrm{Tor}^R_i(M, N) = 0$ *for some even $i > \nu$ and* $\mathrm{Tor}^R_j(M, N) = 0$ *for some odd $j > \nu$, then* $\mathrm{Tor}^R_n(M, N) = 0$ *for all $n \geq d - b + 1$.*

The proof of Theorem 3.1 will depend on the following lemma.

LEMMA 3.2. *Let Q be a local ring with infinite residue field and set $R := Q/(\mathbf{x})$, where $\mathbf{x} := x_1, \ldots, x_r$ is a Q-regular sequence. Let B and C be not necessarily finitely generated R-modules. Suppose that* $\mathrm{Ext}^*_Q(B, C)$ *is finitely generated as a Q-module. Then there exists a positive integer ν such that one of the following happens*:

(1) $\mathrm{Ext}^n_R(B, C) \neq 0$ *for all $n \geq \nu$*

(2) $\mathrm{Ext}^{2n}_R(B, C) \neq 0$ *and* $\mathrm{Ext}^{2n+1}_R(B, C) = 0$ *for all $n \geq \nu/2$*

(3) $\mathrm{Ext}^{2n}_R(B, C) = 0$ *and* $\mathrm{Ext}^{2n+1}_R(B, C) \neq 0$ *for all $n \geq \nu/2$*

(4) $\mathrm{Ext}^n_R(B, C) = 0$ *for all $n \geq \nu$.*

Proof. Gulliksen has shown [G] that $\operatorname{Ext}_R^*(B,C)$ can be regarded as a nonnegatively graded module over a polynomial ring $\mathscr{R} := R[\chi_1, \ldots, \chi_r]$ in such a way that $\operatorname{Ext}_R^*(B,C)$ is Noetherian as an $\mathscr{R}$-module whenever $\operatorname{Ext}_Q^*(B,C)$ is Noetherian as a Q-module. Here, the nth homogeneous component $\operatorname{Ext}_R^*(B,C)_n$ of $\operatorname{Ext}_R^*(B,C)$ is simply $\operatorname{Ext}_R^n(B,C)$, and the degree of each χ_i is 2. By this result and our hypothesis, $\operatorname{Ext}_R^*(B,C)$ is a graded Noetherian $\mathscr{R}$-module.

Since the χ_i act in degree 2, we can decompose $\operatorname{Ext}_R^*(B,C)$ as the direct sum of its even and odd parts. Let E denote the graded submodule of even homogeneous components and let O denote the graded submodule of odd homogeneous components.

Recall that a homogeneous element x of a nonnegatively graded ring A is *superficial* for the graded A-module X provided $(0:_X x) \cap X_n = 0$ for all $n \gg 0$. If A is Noetherian with $A = A_0[A_\alpha]$, A_0 is local with infinite residue field, and X is finitely generated, then superficial elements exist for X in degree α. (The proof is a slight generalization of (22.1) of [N].) Therefore we can choose superficial elements χ_e and χ_o of degree 2 for E and O, respectively. The existence of χ_e implies that there cannot simultaneously exist infinitely many zero and infinitely many nonzero homogeneous components of E. The same holds for O. Letting ν be sufficiently large, we see that one of the situations (1)–(4) must occur. ∎

Proof of (3.1). We suppose that $R = Q/(\mathbf{x})$, where Q is a regular local ring and $\mathbf{x} = x_1, \ldots, x_r$ is a Q-regular sequence. We harmlessly assume that the residue field k of R is infinite.

Whenever A is a local ring we let $\mathrm{E}_A(l)$ denote the injective hull of the residue field l of A. Suppose X and Y are A-modules. Then there are isomorphisms

$$\operatorname{Hom}_A\big(\operatorname{Tor}_i^A(X,Y), \mathrm{E}_A(l)\big) \cong \operatorname{Ext}_A^i\big(X, \operatorname{Hom}_A(Y, \mathrm{E}_A(l))\big) \quad (3.1.1)$$

for all i [Rot, p. 360].

First we want to show that $\operatorname{Ext}_Q^*(M, \operatorname{Hom}_R(N, \mathrm{E}_R(k)))$ is finitely generated over Q. Since $M \otimes_R N$ has finite length, $\operatorname{Tor}_i^Q(M,N)$ has finite length for all $i \geq 0$. By duality $\operatorname{Hom}_Q(\operatorname{Tor}_i^Q(M,N), \mathrm{E}_Q(k))$ has finite length for all i [BH, (3.2.12)]. From (3.1.1) we conclude that $\operatorname{Ext}_Q^i(M, \operatorname{Hom}_Q(N, \mathrm{E}_Q(k)))$ has finite length for all i. Observe that $\operatorname{Hom}_Q(R, \mathrm{E}_Q(k)) = (0:_{\mathrm{E}_Q(k)} \mathbf{x})$ is both injective as an R-module and an R-essential extension of k. Hence $\mathrm{E}_R(k) \cong \operatorname{Hom}_Q(R, \mathrm{E}_Q(k))$, and by Hom-tensor adjointness [Rot, p. 363] we have the R-module isomorphism $\operatorname{Hom}_R(N, \mathrm{E}_R(k)) \cong \operatorname{Hom}_Q(N, \mathrm{E}_Q(k))$. Thus $\operatorname{Ext}_Q^i(M, \operatorname{Hom}_R(N, \mathrm{E}_R(k)))$ has finite length for all i. Finally, since $\operatorname{pd}_Q M < \infty$, we see that $\operatorname{Ext}_Q^*(M, \operatorname{Hom}_R(N, \mathrm{E}_R(k)))$ is indeed finitely generated over Q. (In fact, it has finite length.)

Now we can apply Lemma 3.2 with $B = M$ and $C = \operatorname{Hom}_R(N, \mathrm{E}_R(k))$. Let ν be the integer obtained from the conclusion of Lemma 3.2. We want to show that ν has the desired property: suppose that $\operatorname{Tor}_i^R(M, N) = 0$ for some even $i > \nu$ and $\operatorname{Tor}_j^R(M, N) = 0$ for some odd $j > \nu$. Then we have

$$\operatorname{Ext}_R^i\big(M, \operatorname{Hom}_R(N, \mathrm{E}_R(k))\big) \cong \operatorname{Hom}_R\big(\operatorname{Tor}_i^R(M, N), \mathrm{E}_R(k)\big) = 0,$$

and (similarly) $\operatorname{Ext}_R^j(M, \operatorname{Hom}_R(N, \mathrm{E}_R(k))) = 0$ for i even and j odd $> \nu$. Hence, by Lemma 3.2,

$$\operatorname{Hom}_R\big(\operatorname{Tor}_n^R(M, N), \mathrm{E}_R(k)\big) \cong \operatorname{Ext}_R^n\big(M, \operatorname{Hom}_R(N, \mathrm{E}_R(k))\big) = 0$$

for all $n \geq \nu$. This means that $\operatorname{Tor}_n^R(M, N) = 0$ for all $n \geq \nu$. Applying Proposition 2.2, we get the desired conclusion. ∎

4. EXAMPLES

In this section we give two examples. The first (Example 4.1) shows that Proposition 2.3 is sharp in the sense that one cannot, in general, get by with fewer consecutive vanishing Tors in the hypothesis of a rigidity theorem. The second (Example 4.2) shows that the necessary condition for the vanishing of all higher Tors of Huneke and Wiegand—that one of the modules must be of finite projective dimension—does not extend to complete intersections of codimension greater than 1.

EXAMPLE 4.1. Let n be a positive integer and

$$R := k[[X_1, \dots, X_n, Y_1, \dots, Y_n]]/(X_1Y_1, \dots, X_nY_n),$$

where k is a field and the X_i and Y_i are analytic indeterminates. (Note that R is a complete intersection of dimension n and codimension n.) Set $M := R/(x_1, \dots, x_n)$, and let $(\mathbf{F}, \partial)$ be a minimal R-free resolution of M. Given $s \geq 0$, define $M_s := \operatorname{coker}(F^*_{n+s-1} \xrightarrow{\partial^*_{n+s}} F^*_{n+s})$, where $(\ )^* := \operatorname{Hom}_R(\_, R)$. Finally, let $N := R/(y_1, \dots, y_n)$. Then both M_s and N have complexity n, and

$$\operatorname{Tor}^R_{1+s}(M_s, N) = \cdots = \operatorname{Tor}^R_{n+s}(M_s, N) = 0,$$

whereas $\operatorname{Tor}_S^R(M_s, N) \neq 0$ and $\operatorname{Tor}^R_{n+s+1}(M_s, N) \cong k$.

These assertions were initially established by the author for $n = 2, 3$ by direct computation and conjectured to hold for all n. The general argument below is due to L. Avramov.

Proof. We first observe that $y_1, \dots, y_n$ is a regular sequence on M and $x_1, \dots, x_n$ is a regular sequence on N. Hence M and N are maximal Cohen–Macaulay R-modules. The fact that M is a maximal Cohen–Macaulay module means that

$$\operatorname{Ext}_R^i(M, R) = 0 \qquad \text{for all } i > 0, \tag{4.1.1}$$

and the fact that $x_1, \dots, x_n$ is N-regular implies

$$\operatorname{Ext}_R^i(M, N) = 0 \quad \text{for } i < n \qquad \text{and} \qquad \operatorname{Ext}_R^n(M, N) \neq 0. \tag{4.1.2}$$

Property (4.1.1) implies that the dual

$$0 \to M^* \to F_0^* \xrightarrow{\partial_1^*} F_1^* \to \cdots \tag{4.1.3}$$

of the undeleted free resolution of M

$$\cdots \to F_1 \xrightarrow{\partial_1} F_0 \to M \to 0 \tag{4.1.4}$$

is again exact. It is easy to see that $M^* \cong (y_1 \cdots y_n) \cong M$. Hence we can splice together (4.1.3) and (4.1.4), getting a doubly infinite exact sequence of free modules, where we write the degrees of the free modules beneath them:

$$\mathbf{G}\colon \cdots \to \underset{2}{F_2} \xrightarrow{\partial_2} \underset{1}{F_1} \xrightarrow{\partial_1} \underset{0}{F_0} \xrightarrow{[y_1 \cdots y_n]} \underset{-1}{F_0^*} \xrightarrow{\partial_1^*} \underset{-2}{F_1^*} \xrightarrow{\partial_2^*} \underset{-3}{F_2^*} \to \cdots.$$

We see that the truncation $\mathbf{G}_{\geq -n-s-1}$ is a free resolution of $M_s := \operatorname{coker}(F_{n+s-1}^* \xrightarrow{\partial_{n+s}^*} F_{n+s}^*)$. Now that we have a truncated complex $\mathbf{G}_{\geq -n-s-1}$, we switch to the normal degree conventions. That is, $(G_{\geq -n-s-1})_0 = G_{-n-s-1}$, $(G_{\geq -n-s-1})_1 = G_{-n-s}$, etc.

For all $i \geq 0$ we have a commutative diagram:

$$\begin{array}{ccccc}
F_{i-1}^* \otimes_R N & \xrightarrow{\partial_i^* \otimes \mathrm{N}} & F_i^* \otimes_R N & \xrightarrow{\partial_{i+1}^* \otimes \mathrm{N}} & F_{i+1}^* \otimes_R N \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Hom}_R(F_{i-1}, N) & \xrightarrow{\operatorname{Hom}_R(\partial_i, N)} & \operatorname{Hom}_R(F_i, N) & \xrightarrow{\operatorname{Hom}_R(\partial_{i+1}, N)} & \operatorname{Hom}_R(F_{i+1}, N),
\end{array}$$

where the vertical arrows are natural isomorphisms. (We take F_{-1} to be 0.) Hence, for all $i (0 \leq i \leq n + s - 1)$, we get

$$\operatorname{Tor}_{n+s-i}^R(M_s, N) \cong \operatorname{Ext}_R^i(M, N).$$

Thus, by (4.1.2), $\operatorname{Tor}_i^R(M_s, N) = 0$ for all $i = 1 + s, \dots, n + s$, and $\operatorname{Tor}_s^R(M_s, N) \neq 0$.

Identify N with $A := k[[X_1, \ldots, X_n]]$ and consider the $(n+s+1)$st degree piece of $\mathbf{G}_{\geq -n-s-1} \otimes_R N$:

$$A^n \xrightarrow{\partial_1 \otimes N} \underset{n+s+1}{A} \xrightarrow{0} A.$$

It is clear that $\operatorname{im}(\partial_1 \otimes N) = (X_1, \ldots, X_n)A$, and so $\operatorname{Tor}^R_{n+s+1}(M_s, N) \cong k$. This establishes the claim about the Tors.

Concerning the complexity of M_s and N, Tate [T] gives a free resolution $\mathbf{F}$ of M in his Theorem 4 as the tensor of the divided power algebra $\mathbf{D}(R^n) :=^* \operatorname{Hom}(R[t_1, \ldots, t_n], R)$ $(\deg t_i = -2)$ with the exterior algebra $\wedge Q^n$. Moreover, since $(X_1Y_1, \ldots, X_nY_n) \subseteq \mathbf{n} \cdot \operatorname{ann}_Q M$ $(\mathbf{n} := (X_1, \ldots, X_n, Y_1, \ldots, Y_n))$, this resolution $\mathbf{F}$ is minimal (see, e.g., (6.3) of [A3]). Also in [T], Tate shows that the Poincaré series $P_R^M(t) := \sum_{i \geq 0} b_i^R(M)t^i$ is equal to $(1+t)^n/(1-t^2)^n = 1/(1-t)^n$. Hence the Betti numbers $b_i^R(M)$ of M are eventually given by a polynomial degree $n-1$. That is, $\operatorname{cx}_R M = n$. By symmetry, $\operatorname{cx}_R N = n$. Finally, since M is a syzygy of M_s, the complexity of M_s is also n. ▮

Remarks. (1) Looking at what is used in the proof, one could obviously generalize Example 4.1 to specific pairs of modules over local rings more general than quotients of power series rings over a field.

(2) One could also construct examples like the one above, in which $r - \operatorname{cx}_R M = r - \operatorname{cx}_R N$ is any positive integer (where r denotes the codimension of R).

(3) In Example 4.1, $M_S \otimes_R N$ does not have finite length. It would be interesting to know whether examples such as Example 4.1—of n consecutive vanishing Tors without subsequent vanishing (with arbitrary positive indices)—exist with $M_s \otimes_R N$ having finite length. This would show there is no bound on the ν of Theorem 3.1.

EXAMPLE 4.2. Let $R = k[[X, Y, Z]]/(XZ - Y^2, XY - Z^2)$, $M = R/(x, y)$, and $N = R/(x, z)$. Then R is a complete intersection of codimension 2 and $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \gg 0$, but neither M nor N has finite projective dimension over R.

Proof. A minimal free resolution of M is

$$\mathbf{F}\colon \cdots \to R^2 \xrightarrow{\begin{bmatrix} y & z \\ x & y \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} -y & z \\ x & -y \end{bmatrix}} R^2 \xrightarrow{[x \quad y]} R.$$

Similarly, any minimal free resolution of N is infinite and periodic of period 2 after the first stage. To compute $\operatorname{Tor}_i^R(M, N)$, we tensor $\mathbf{F}$ with N (which amounts to killing (x, z)) and compute homology. We can identify

N with the ring $A := k[[U]]/(U^2)$ via $u \leftrightarrow y$. Hence $\mathbf{F} \otimes_R N$ is really just

$$\mathbf{G}\colon \cdots A^2 \xrightarrow{\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}} A^2 \xrightarrow{\begin{bmatrix} -u & 0 \\ 0 & -u \end{bmatrix}} A^2 \xrightarrow{[0 \quad u]} A.$$

It is easy to see that $\mathbf{G}$ is exact at the second stage and beyond. Thus $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 2$. ∎

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